Master of Science (Mathematics)
Semester – II

Paper Code –

INTEGRAL EQUATIONS AND CALCULUS OF VARIATIONS

Paper Code :

Integral Equations and Calculus of Variations

M. Marks = 100
Term End Examination = 80
Assignment = 20

Time = 3 Hours

Course Outcomes

Students would be able to:

CO1 Understand the methods to reduce Initial value problems associated with linear differential equations to various integral equations.

CO2 Categorise and solve different integral equations using various techniques.

CO3 Describe importance of Green's function method for solving boundary value problems associated with non-homogeneous ordinary and partial differential equations, especially the Sturm-Liouville boundary value problems.

CO4 Learn methods to solve various mathematical and physical problems using variational techniques.

Section - I

Linear Integral equations, Some basic identities, Initial value problems reduced to Volterra integral equations, Methods of successive substitution and successive approximation to solve Volterra integral equations of second kind, Iterated kernels and Neumann series for Volterra equations. Resolvent kernel as a series. Laplace transfrom method for a difference kernel. Solution of a Volterra integral equation of the first kind.

Section - II

Boundary value problems reduced to Fredholm integral equations, Methods of successive approximation and successive substitution to solve Fredholm equations of second kind, Iterated kernels and Neumann series for Fredholm equations. Resolvent kernel as a sum of series. Fredholmresolvent kernel as a ratio of two series. Fredholm equations with separable kernels. Approximation of a kernel by a separable kernel, Fredholm Alternative, Non homonogenous Fredholm equations with degenerate kernels.

Section - III

Green function, Use of method of variation of parameters to construct the Green function for a nonhomogeneous linear second order boundary value problem, Basic four properties of the Green function, Alternate procedure for construction of the Green function by using its basic four properties. Reduction of a boundary value problem to a Fredholm integral equation with kernel as Green function, Hilbert-Schmidt theory for symmetric kernels.

Section - IV

Motivating problems of calculus of variations, Shortest distance, Minimum surface of resolution, Brachistochrone problem, Isoperimetric problem, Geodesic. Fundamental lemma of calculus of variations, Euler equation for one dependant function and its generalization to 'n' dependant functions and to higher order derivatives. Conditional extremum under geometric constraints and under integral constraints.

Note : The question paper of each course will consist of **five** Sections. Each of the sections **I to IV** will contain **two** questions and the students shall be asked to attempt **one** question from each. **Section-V** shall be **compulsory** and will contain **eight** short answer type questions without any internal choice covering the entire syllabus.

Books Recommended:

1. Jerri, A.J., Introduction to Integral Equations with Applications, A Wiley-Interscience Publication, 1999.
2. Kanwal, R.P., Linear Integral Equations, Theory and Techniques, Academic Press, New York.
3. Lovitt, W.V., Linear Integral Equations, McGraw Hill, New York.
4. Hilderbrand, F.B., Methods of Applied Mathematics, Dover Publications.
5. Gelfand, J.M., Fomin, S.V., Calculus of Variations, Prentice Hall, New Jersey, 1963.

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Volterra Integral Equations

Structure

1.1. Introduction.

1.2. Integral Equation.

1.3. Solution of Volterra Integral Equation.

1.4. Laplace transform method to solve an integral equation.

1.5. Solution of Volterra Integral Equation of first kind.

1.6. Method of Iterated kernel/Resolvent kernel to solve the Volterra integral equation.

1.7. Summary

1.1. Introduction. This chapter contains basic definitions and identities for integral equations, various methods to solve Volterra integral equations of first and second kind. Iterated kernels and Neumann series for Volterra equations.

1.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

i. Initial value problem reduced to Volterra integral equations.

ii. Method of successive substitution to solve Volterra integral equation of second kind.

iii. Method of successive approximation to solve Volterra integral equation of second kind.

iv. Resolved kernel as a series.

v. Laplace transform method for a difference kernel.

1.1.2. Keywords. Integral Equations, Volterra Integral Equations, Iterated Kernels.

1.2. Integral Equation. An integral equation is one in which function to be determined appears under the integral sign. The most general form of a linear integral equation is

$$h(x)\, u(x) = f(x) + \int_a^{b(x)} K(x\,,\ \xi)\, u(\xi)\, d\xi \quad \text{for all } x \in [a, b]$$

in which, u(x) is the function to be determined and **K(x, ξ) is called the Kernel of integral equation**.

1.2.1. Volterra Integral equation. A Volterra integral equation is of the type:

$$h(x)\, u(x) = f(x) + \int_a^{x} K(x\,,\ \xi)\, u(\xi)\, d\xi \quad \text{for all } x \in [a, b]$$

that is, in Volterra equation b(x) = x

(i) If h(x) = 0, the above equation reduces to

$$-\, f(x) = \int_a^{x} K(x\,,\ \xi)\, u(\xi)\, d\xi$$

This equation is called **Volterra integral equation of first kind**.

(ii) If h(x) = 1, the above equation reduces to

$$u(x) = f(x) + \int_a^{x} K(x\,,\ \xi)\, u(\xi)\, d\xi$$

This equation is called **Volterra integral equation of second kind**.

1.2.2. Homogeneous integral equation. If f(x) = 0 for all x $\in$ [a, b], then the reduced equation

$$h(x)\, u(x) = \int_a^{bx} K(x\,,\ \xi)\, u(\xi)\, d\xi$$

is called homogeneous integral equation. Otherwise, it is called non-homogeneous integral equation.

1.2.3. Leibnitz Rule. The Leibnitz rule for differentiation under integral sign:

$$\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} F(x\,,\ \xi)\, d\xi\right] = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x} d\xi + F(x\,,\ \beta(x))\ \frac{d\beta(x)}{dx}\ - F(x\,,\ \alpha(x))\frac{d\alpha(x)}{dx}$$

In particular, we have

$$\frac{d}{dx}\left[\int_a^{x} K(x\,,\ \xi) u(\xi)\, d\xi\right] = \int_a^{x} \frac{\partial K}{\partial x} u(\xi)\, d\xi + K(x\,,\ x)\, u(x)\ .$$

1.2.4. Lemma. If n is a positive integer, then

$$\int_a^x\int_a^{x_1}\cdots\int_a^{x_{n-2}}\int_a^{x_{n-1}} F(x_n)\, dx_n\; dx_{n-1}\ldots dx_1 = \frac{1}{n-1!}\int_a^x (x-\xi)^{n-1} f(\xi)\, d\xi\,.$$

Proof. If $I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)\, d\xi$, then $I_n(a) = 0$ and for n = 1, $I_1(x) = \int_a^x f(\xi)\, d\xi$.

Using Leibnitz rule, we get $\frac{dI_1}{dx}$ = f(x).

Now, differentiating $I_n(x)$ w.r.t. x and using Leibnitz rule, we get

$$\frac{dI_1}{dx} = \frac{\mathrm{d}}{\mathrm{dx}}.\;\; . = \int_a^x \frac{\partial}{\partial x}\left[(x-\xi)^{n-1}\right] f(\xi)\, d\xi = (\mathrm{n}-1)\int_a^x (x-\xi)^{n-2} f(\xi)\, d\xi$$

or

$$\frac{dI_n(x)}{dx} = (\mathrm{n}-1)\; I_{n-1}(x) \text{ for n > 1}$$

Taking successive derivatives, we get

$$\frac{d^{n-1}}{dx^{n-1}}\, I_n(x) = (n-1)\,(n-2)\ldots 2.1\; I_1(x)$$

Again, differentiating,

$$\frac{d^n}{dx^n} I_n(x) = n-1!\; \frac{d}{dx}\, I_{1(x)} = n-1!\; f(x) \qquad (1)$$

We observe that,

$$I_n^{(m)}\,(a) = 0 \text{ for m = 0, 1, 2,…, n } -1 \qquad (2)$$

Integrating (1) over the interval [a, x] and using (2) for m = n -1, we obtain

$$I_n^{(n-1)}\,(x) = (n-1)!\int_a^x f(x_1)\, dx_1$$

Again integrating it and using (2) for m = n -2, we get

$$\frac{d^{n-2}}{dx^{n-2}} I_n(x) = I_n^{(n-2)}(x) = n-1!\int_a^x\int_a^{x_1} f(x_2)\, dx_2\; dx_1$$

Continuing like this, n times, we obtain

$$I_n(x) = (n-1)!\int_a^x\int_a^{x_1}\ldots\ldots\int_a^{x_{n-1}} f(x_n)\, dx_n\, dx_{n-1}\ldots\ldots dx_1\,.$$

which provides the required result.

1.2.5. Example. Transform the initial value equation $\frac{d^2y}{dx^2}$ + x $\frac{dy}{dx}$+y = 0; y(0) = 1, $y'(0) = 0$ to Volterra integral equation.

Solution. Consider,
$$\frac{d^2y}{dx^2} = \phi(x) \qquad (1)$$

Then
$$\frac{dy}{dx} = \int_0^x \phi(\xi)\, d\xi + C_1$$

Using the condition $y'(0) = 0$, we get, $C_1 = 0$
$$\frac{dy}{dx} = \int_0^x \phi(\xi)\, d\xi \qquad (2)$$

Again, integrating from 0 to x and using the above lemma, we get
$$y = \int_0^x (x-\xi)\ \phi(\xi)\, d\xi + C_2$$

Using the condition y(0) = 1, we get $C_2 = 1$

So,
$$y = \int_0^x (x-\xi)\ \phi(\xi)\, d\xi + 1 \qquad (3)$$

From the relations (1), (2) and (3), the given differential equation reduces to :
$$\phi(x) + x\int_0^x \phi(\xi)\, d\xi + \int_0^x (x-\xi)\ \phi(\xi)\, d\xi + 1 = 0$$

or
$$\phi(x) = -1 - \int_0^x (2x-\xi)^{n-1}\phi(\xi)\, d\xi$$

which represents a Volterra integral equation of second kind.

1.2.6. Exercise. Reduce following initial value problem into Volterra integral equations:

1. y″ + xy = 1, $y'(0) = 0 = y(0)$.

Answer. $y(x) = \frac{x^2}{2} - \int_0^x (x-\xi)\ \xi\ y(\xi)\, d\xi$.

2. $\frac{d^2y}{dx^2}$ +A(x)$\frac{dy}{dx}$ + B(x)y = g(x), y(a) = c_1 and y′(a) = c_2.

Answer. f(x) = $c_1 + c_2(x-a) + \int_a^x (x-\xi)\, g(\xi)\, d\xi + A(a)\, c_1(x-a)$,

where $K(x,\xi) = (x-\xi)\left[A'(\xi)-B(\xi)\right]-A(\xi)$.

3. $y'' + \lambda y = 0,\ y(0) = 1,\ y'(0) = 0$.

Answer. $y(x) = 1 - \lambda \int_0^x (x-\xi)\ y(\xi)\, d\xi$.

4. $y'' - 5y' + 6y = 0,\ y(0) = 0,\ y'(0) = -1$.

Answer. $y(x) = (6x-5) + \int_0^x (5-6x+6\xi)\, \phi(\xi)\, d\xi$.

1.3. Solution of Volterra Integral Equation.

1.3.1. Weierstrass M-Test. Suppose $\sum f_n(z)$ is an infinite series of single valued functions defined in a bounded closed domain D. Let $\sum M_n$ be a series of positive constants (independent of z) such that

(i) $|f_n(z)| \le M_n$ for all n and for all $z \in D$.

(ii) $\sum M_n$ is convergent.

Then the series $\sum f_n$ is uniformly and absolutely convergent in D.

1.3.2. Theorem. Let $u(x) = f(x) + \lambda \int_a^x K(x,\ \xi)\, u(\xi)\, d\xi$ be a non-homogeneous Volterra integral equation of second kind with constants a and λ. f(x) is a non-zero real valued continuous function in the interval I = [a, b]. $K(x,\ \xi)$ is a non-zero real valued continuous function defined in the rectangle $R = I \times I = \{(x,\ \xi) : a \le x,\ \xi \le b\}$ and $|K(x,\ \xi)| \le M$ in R.

Then the given equation has one and only one continuous solution u(x) in I and this solution is given by the absolutely and uniformly convergent series.

$$u(x) = f(x) + \lambda \int_a^x K(x,\ t)\, f(t)\, dt + \lambda^2 \int_a^x \int_a^t K(x,\ t)\, K(t,\ t_1)\, f(t_1)\, dt_1 dt + \ldots.$$

Proof. This theorem can be proved by applying either of the following two methods :

(a) Method of successive substitution.

(b) Method of successive approximation.

Let us apply these methods one by one.

(a) Method of Successive Substitution. The given integral equation is

$$u(x) = f(x) + \lambda \int_a^x K(x,\ t)\ u(t)\ dt \tag{1}$$

Substituting value of u(t) from (1) into itself, we get

$$u(x) = f(x) + \lambda\ . \int_a^x K(x,\ t) \left[f(t) + \lambda \int_a^t K(t,\ t_1)\ u(t_1)\ dt_1 \right] dt\ .$$

$$= f(x)\ \ \lambda \int_a^x K(x,\ t)\ f(t)\ dt + \lambda^2 \int_a^x \int_a^t K(x,\ t)\ K(t,\ t_1)\ u(t_1)\ dt_1 dt \tag{2}$$

Again substituting the value of $u(t_1)$ from (1) into (2), we get

$$u(x) = f(x) + \lambda \int_a^x K(x,\ t)\ f(t)\ dt + \lambda^2 \int_a^x \int_a^t K(x,\ t)\ K(t,\ t_1)\ f(t_1)\ dt_1 dt$$

$$+ \lambda^3 \int_a^x \int_a^t \int_a^{t_1} K(x,\ t)\ K(t,\ t_1) K(t_1,\ t_2)\ u(t_2)\ dt_2 dt_1 dt$$

Proceeding in the same way, we get after n steps

$$\begin{aligned} u(x) = {} & f(x) + \lambda \int_a^x K(x\ ,\ t)\ f(t) dt \\ & + \ldots + \lambda^n \int_a^x \int_a^t \ldots \int_a^{t_{n-2}} K(x,\ t)\ K(t,\ t_1) \ldots K(t_{n-2}, t_{n-1})\ f(t_{n-1}) dt_{n-1}\ dt_{n-2} \ldots . dt_1 dt + R_{n+1}(x) \end{aligned} \tag{3}$$

where $R_{n+1}(x) = \lambda^{n+1} \int_a^x \int_a^t \ldots \int_a^{t_{n-1}} K(x,\ t)\ K(t,\ t_1) \ldots K(t_{n-1}, t_n)\ u(t_n) dt_n dt_{n-1} \ldots dt_1 dt$ (4)

Consider the infinite series,

$$f(x) + \lambda \int_a^x K(x,\ t)\ f(t) dt\ + \lambda^2 \int_a^x \int_a^t K(x,\ t)\ K(t,\ t_1)\ f(t_1)\ dt_1 dt + \ldots \tag{5}$$

Neglecting the first term, let $v_n(x)$ denotes the nth term of infinite series in (5). Since f(x) is continuous over I, so it is bounded.

Let $|f(x)| \le N$ in I. Also, it is given that $|K(x,\ t)| \le M$ in R. Therefore,

$$|v_n(x)| \le |\lambda|^n \int_a^x \int_a^t \ldots \int_a^{t_{n-2}} M^n \; N \; dt_{n-1} \ldots dt_1 \; dt$$

Thus, $$|v_n(x)| \le |\lambda|^n M^n \; N \; \frac{(x-a)^n}{n!} \le |\lambda|^n M^n \; N \; \frac{(b-a)^n}{n!} \qquad (6)$$

The series whose nth term is $|\lambda|^n M^n \; N \; \dfrac{(b-a)^n}{n!}$ is a series of positive terms and is convergent by ratio test for all values of a, b, $|\lambda|$, M and N.

Thus, by Weierstrass M–test, the series $\sum v_n(x)$ is absolutely and uniformly convergent in I .

If u(x) given by (2) is continuous in I, then is bounded in I, that is,

u(x) ≤ U for all x in I (7)

Then, $|R_{n+1}(x)| \le |\lambda|^{n+1} M^{n+1} u \dfrac{(x-a)^{n+1}}{(n+1)!} \le |\lambda|^{n+1} M^{n+1} u \dfrac{(b-a)^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$

$$\Rightarrow \qquad \lim_{n \to \infty} R_{n+1}(x) = 0 \qquad (8)$$

From equations (3), (4) and (8), we obtain

$$u(x) = f(x) + \lambda \int_a^x K(x, \; t) \, f(t) \; dt \; + \lambda^2 \int_a^x \int_a^t K(x, \; t) \; K(t, \; t_1) \; f(t_1) \; dt_1 \, dt + \ldots \text{to } \infty$$

which is the required series.

Now, we verify that this series is actually a solution of the given Volterra integral (1). Substituting the series for u(x) in the R.H.S. of the given equation, we get

$$\text{R.H.S.} = f(x) + \lambda \int_a^x K(x, \; \xi) \left[f(\xi) + \lambda \int_a^\xi K(\xi, \; t) \, f(t) \; dt + \lambda^2 \int_a^\xi \int_a^t K(\xi, \; t) \; K(t, \; t_1) \; f(t_1) \; dt_1 dt + \ldots \text{to } \infty \right]$$

$$= f(x) + \lambda \int_a^x K(x, \; \xi) \, f(\xi) \; d\xi + \lambda^2 \int_a^x \int_a^\xi K(x, \; \xi) \; K(\xi, \; t) \; f(t) \; dt \; d\xi + \ldots \text{to } \infty = u(x) = \text{L.H.S.}$$

(b) Method of Successive Approximation. In this method, we select any real valued function, say $u_0(x)$, continuous on I = [a, b] as the zeroth approximation. Substituting this zeroth approximation in the given Volterra integral equation.

$$u(x) = f(x) + \lambda \int_a^x K(x, \; t) \; u(t) \; dt \qquad (1)$$

We obtain the first approximation, say $u_1(x)$, given by

$$u_1(x) = f(x) + \lambda \int_a^x K(x\ ,\ t)\ u_0(t)\ dt \tag{2}$$

The value of $u_1(x)$ is again substituted for $u(x)$ in (1) to obtain the second approximation, $u_2(x)$ where

$$u_2(x) = f(x) + \lambda \int_a^x K(x\ ,\ t)\ u_1(t)\ dt \tag{3}$$

This process is continued to obtain n^{th} approximation

$$u_n(x) = f(x) + \lambda \int_a^x K(x\ ,\ t)\ u_{n-1}(t)\ dt \quad \text{for } n = 1, 2, 3,\ldots \tag{4}$$

This relation is known as recurrence relation.

Now, we can write

$$u_n(x) = f(x) + \lambda \int_a^x K(x,\ t) \left[f(t) + \lambda \int_a^t K(t,\ t_1)\ u_{n-2}(t_1)\ dt_1 \right] dt$$

$$= f(x) + \lambda \int_a^x K(x,\ t)\ f(t)\ dt + \lambda^2 \int_a^x \int_a^t K(x,\ t)\ K(t,\ t_1) \left[f(t_1) + \lambda \int_a^{t_1} K(t_1,\ t_2) u_{n-3}(t_2)\ dt_2 \right] dt_1\ dt$$

or $$u_n(x) = f(x) + \lambda \int_a^x K(x,\ t)\ f(t)\ dt + \lambda^2 \int_a^x \int_a^t K(x,\ t)\ K(t,\ t_1) f(t_1) dt_1 dt$$

$$+\lambda^3 \int_a^x \int_a^t \int_a^{t_1} K(x,\ t)\ K(t,\ t_1)\, K(t_1,\ t_2) dt_2\ dt_1 dt \tag{5}$$

Continuing in this fashion, we get

$$u_n(x) = f(x) + \lambda \int_a^x K(x,\ t)\ f(t)\ dt + \ldots$$

$$+\lambda^{n-1} \int_a^x \int_a^t \ldots \int_a^{t_{n-3}} K(x,\ t)\ K(t,\ t_1)\ldots K(t_{n-3},\ t_{n-2})\, f(t_{n-2}) dt_{n-2} \ldots dt_1 dt + R_n(x)\mathbb{F} \tag{6}$$

where

$$R_n(x) = \lambda^n \int_a^x \int_a^t \ldots \int_a^{t_{n-2}} K(x,\ t)\ K(t,\ t_1) \ldots K(t_{n-2},\ t_{n-1})\, u_0\,(t_{n-1}) dt_{n-1} \ldots dt_1 dt \qquad (7)$$

Since $u_0(x)$ is continuous on I so it is bounded.

Let $$|u_0(x)| \ \leq\ u \ \text{ in } \ I \qquad (8)$$

Thus, $|R_n(x)| \leq |\lambda|^n M^n u \dfrac{(x-a)^n}{n!} \leq |\lambda|^n M^n u \dfrac{(b-a)^n}{n!} \to 0$ as $n \to \infty$

So, $$\lim_{n \to \infty} R_n(x) = 0. \qquad (9)$$

Thus, as n increases, the sequence < $u_n(x)$ > approaches to a limit. We denote this limit by u(x) that is,

$$u(x) = \lim_{n\to\infty} u_n(x)$$

So, $$u(x) = f(x) + \lambda \int_a^x K(x,\ t)\, f(t)\ dt \ + \lambda^2 \int_a^x \int_a^t K(x,\ t)\ K(t,\ t_1)\ f(t_1)\ dt_1 dt + \ldots \text{to } \infty \qquad (10)$$

As in the method of successive substitution, we can prove that the series (10) is absolutely and uniformly convergent and hence the series on R.H.S. of (10) is the desired solution of given Volterra integral equation.

Uniqueness. Let, if possible, the given Volterra integral equation has another solution v(x). We make, by our choice, the zeroth approximation $u_0(x) = v(x)$, then all approximations $u_1(x),\ldots,u_n(x)$ will be identical with v(x) that is,

$$u_n(x) = v(x) \text{ for all n}$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n(x) = v(x)$$

$$\Rightarrow \quad u(x) = v(x)$$

This proves uniqueness of solution. With this, the proof of the theorem is completed.

1.3.3. Example. Using the method of successive approximation solve the integral equation,

$$u(x) = x - \int_0^x (x-\xi)\ u(\xi)\ d\xi \qquad (1)$$

Solution. Let the zeroth approximation be $u_0(x) = 0$

Then the first approximation $u_1(x)$ is given by :

$$u_1(x) = x - \int_0^x 0.\ d\xi \ = x \qquad (2)$$

Thus, $u_2(x) = x - \int_0^x (x-\xi)\, u_1(\xi)\, d\xi = x - \int_0^x (x-\xi)\, \xi\, d\xi$

$$= x - \left[\frac{x\xi^2}{2}\right]_0^x + \left[\frac{\xi^3}{3}\right]_0^x = x - \frac{x^3}{2} + \frac{x^3}{3}$$

$$= x - \frac{x^3}{6} = x - \frac{x^3}{3!} \qquad (3)$$

Now, $u_3(x) = x - \int_0^x (x-\xi)\, u_2(\xi)\, d\xi$

$$= x - \int_0^x (x-\xi) \left(\xi - \frac{\xi^3}{6}\right) d\xi$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad (4)$$

From (2), (3) and (4), we conclude that the nth approximation, $u_n(x)$ will be

$$u_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \qquad (5)$$

which is obviously the nth partial sum of Maclaurin's series of sinx. Hence by the method of successive approximation, solution of given integral equation is

$$u(x) = \lim_{n\to\infty} u_n(x) = \sin x$$

Hence the solution.

1.3.4. Exercise.

1. Using the method of successive approximation, solve the integral equation,

$$y(x) = e^x + \int_0^x e^{x-t}\, y(t)\, dt .$$

Answer. $y(x) = \lim_{n\to\infty} e^x \left[1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}\right] = e^x . e^x = e^{2x}$.

2. $u(x) = 1 + \int_0^x (x-\xi)\, u(\xi)\, d\xi$.

Answer. cosh x

3. $u(x) = 1 + \int_0^x (\xi - x)\, u(\xi)\, d\xi$

Answer. cos x

4. $u(x) = 1 + \int_0^x u(\xi)\, d\xi$

Answer. e^x

5. $u(x) = e^{x^2} + \int_0^x e^{x^2 - t^2}\, u(t)\, dt$

Answer. $e^{x\,(x+1)}$

6. $u(x) = (1 + x) + \int_0^x (x - \xi)\, u(\xi)\, d\xi$ with $u_0(x) = 1$

Answer. e^x

1.4. Laplace transform method to solve an integral equation.

1.4.1. Definition. The Laplace transform of a function f(x) defined on interval $(0, \infty)$ is given by

$$L[f(x)] = f(s) = \int_0^\infty f\;(x)\, e^{-sx} dx \qquad (1)$$

Here, s is called Laplace variable or Laplace parameter. Also . $f(x) = L^{-1}[f(s)]$ is called inverse Laplace transform.

1.4.2. Some important results.

(1) $L(\sin x) = \frac{1}{s^2+1}$ (2) $L[\cos x] = \frac{s}{s^2+1}$ (3) $L[e^{ax}] = \frac{1}{s-a}$

(4) $L[x^n] = \frac{n!}{s^n+1}$, $n \geq 0$ (5) $L[f'(x)] = s\,f(s) - f(0)$ (6) $L[1] = \frac{1}{s}$.

1.4.3. Convolution. The convolution of two functions $f_1(x)$ and $f_2(x)$ is denoted by

$(f_1 * f_2)(x)$ and is defined as $(f_1 * f_2)(x) = \int_0^x f_1(x - \xi)\, f_2(\xi)\, d\xi$

1.4.4. Convolution theorem.(without proof)

Laplace transform of convolution of two functions is equal to the product of their respective Laplace transforms, that is, $[(f_1 * f_2)(x)] = L[f_1(x)] . L[f_2(x)]$.

1.4.5. Difference Integral or Convolution Integral. Consider the integral equation

$$u(x) = f(x) + \int_a^{b(x)} K(x,\xi)\, u(\xi)\, d\xi$$

Let the kernel K(x, ξ) be a function of x $-$ ξ, say g(x $-$ ξ) then the integral equation becomes

$$u(x) = f(x) + \int_a^{b(x)} g(x-\xi)\, u(\xi)\, d\xi$$

In this case, the kernel **K(x, ξ) = g(x − ξ) is called difference kernel** and the corresponding integral is called difference integral or convolution integral.

1.4.6. Working Procedure. Consider the integral equation

$$u(x) = f(x) + \lambda \int_0^{x} K(x,\xi)\, u(\xi)\, d\xi$$

where K(x, ξ) is difference kernel of the type g(x $-$ ξ) then,

$$u(x) = f(x) + \lambda \int_0^{x} g(x-\xi)\, u(\xi)\, d\xi$$

$$\Rightarrow \qquad u(x) = f(x) + \lambda\left[g(x) * u(x)\right]$$

Applying Laplace transform on both sides, we get

$$U(s) = F(s) + \lambda\, G(s)\, U(s)$$

where U(s), F(s) and G(s) represent the Laplace Transform of u(x), f(x) and g(x) respectively.

Then, $$U(s) = \frac{F(s)}{1-\lambda G(s)}$$

Applying inverse Laplace Transform

$$u(x) = L^{-1}\left[\frac{F(s)}{1-\lambda G(s)}\right]$$

Note. Method of Laplace Transform is applicable to those integral equations only where the kernel is difference Kernel.

1.4.7. Example. Use the method of Laplace Transform to solve the integral equation.

$$u(x) = x - \int_0^{x} (x-\xi)\, u(\xi)\, d\xi \qquad (1)$$

Solution. Here $K(x, \xi) = x - \xi = g(x-\xi) \Rightarrow g(x) = x$

Thus, (1) can be written as $u(x) = x - g(x) * u(x)$

Applying Laplace Transform on both sides

$$U(s) = L[x] - L[x]\, U(s)$$

$$= \frac{1}{s^2} - \frac{1}{s^2} U(s)$$

$$\Rightarrow \qquad U(s) = \frac{\frac{1}{s^2}}{1+\frac{1}{s^2}} = \frac{1}{s^2+1}$$

So, $$u(x) = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin x.$$

1.4.8. Exercise. Use the method of Laplace Transform to solve the following integral equations.

(1) $u(x) = 1 + \int_0^x (x-\xi)\, u(\xi)\, d\xi$

Answer. cosh x.

(2) $u(x) = 1 + \int_0^x (\xi - x)\, u(\xi)\, d\xi$

Answer. cos x.

(3) $u(x) = 1 + \int_0^x u(\xi)\, d\xi$

Answer. e^x

1.5. Solution of Volterra Integral Equation of first kind. Consider the non-homogeneous Volterra integral equation of first kind

$$f(x) = \lambda \int_0^x K(x, \xi)\, u(\xi)\, d\xi \qquad (1)$$

Where the kernel $K(x, \xi)$ is the difference Kernel of the type

$$K(x, \xi) = g(x-\xi)$$

Then (1) can be written as

$$f(x) = \lambda\, g(x) * u(x)$$

Applying Laplace Transform on both sides :

$$F(s) = \lambda \; G(s)\, U(s)$$

$$\Rightarrow \qquad U(s) = \frac{1}{\lambda}\;\frac{F(s)}{G(s)}$$

Applying inverse Laplace Transform on both sides :

$$u(x) = \frac{1}{\lambda}\; L^{-1}\left[\frac{F(s)}{G(s)}\right]$$

1.5.1. Example. Solve the integral equation sin x = $\lambda\int_0^x e^{x-\xi}\; u(\xi)\; d\xi$ (1)

Solution. Here $K(x, \xi) = e^{x-\xi} = g(x-\xi)$

$$\Rightarrow \qquad g(x) = e^x$$

Thus, (1) can be written as

$$\sin x = \lambda \; g(x) * u(x)$$

Applying Laplace Transform on both sides

$$L\,[\sin x] = \lambda \; L[e^x]\; L[u(x)]$$

$$\Rightarrow \qquad \frac{1}{s^2+1} = \frac{\lambda}{s-1}\; U(s)$$

$$\Rightarrow \qquad U(s) = \frac{1}{\lambda}\;\frac{s-1}{s^2+1} = \frac{1}{\lambda}\;\left[\frac{s}{s^2+1}-\frac{1}{s^2+1}\right]$$

So,

$$u(x) = \frac{1}{\lambda}\; L^{-1}\left[\frac{s}{s^2+1}-\frac{1}{s^2+1}\right]$$

$$u(x) = \frac{1}{\lambda}\;(\cos x - \sin x).$$

1.5.2. Exercise. Solve the integral equation x = $\int_0^x \cos(x-\xi)\; u(\xi)\; d\xi$.

Answer. $1+\frac{x^2}{2}$.

1.5.3. Theorem. Prove that the Volterra integral equation of first kind f(x) = $\lambda\int_0^x K(x\,,\xi)\; u(\xi)\; d\xi$ can be transformed to a Volterra integral equation of second kind, provided that K(x, x) $\neq$ 0.

Proof. The given equation is

$$f(x) = \lambda\int_0^x K(x,\xi)\, u(\xi)\, d\xi \qquad (1)$$

Differentiating (1), w.r.t. x and using Leibnitz rule, we obtain

$$\frac{df}{dx} = \lambda\int_0^x \frac{\partial K}{\partial x} u(\xi)\, d\xi + \lambda K(x\,,\,x)\, u(x).1$$

$$\Rightarrow \qquad -\lambda\, K(x, x)\, u(x) = \lambda\int_0^x \frac{\partial K}{\partial x} u(\xi)\, d\xi - \frac{df}{dx}$$

$$\Rightarrow \qquad u(x) = \frac{1}{\lambda K(x\,,\,x)}.\frac{df}{dx} + \int_0^x -\frac{1}{K(x\,,\,x)}\frac{\partial K}{\partial x} u(\xi)\, d\xi$$

$$\Rightarrow \qquad u(x) = g(x) + \int_0^x H\,(x\,,\,\xi)\, u(\xi)\, d\xi \qquad (*)$$

where $g(x) = \dfrac{1}{\lambda K(x\,,\,x)}\dfrac{df}{dx}$ and $H(x,\ \xi) = \dfrac{-1}{K(x\,,\,x)}\dfrac{\partial K}{\partial x}$. Here, (*) represents the desired Volterra integral equation of second kind.

1.5.4. Example. Reduce the integral equation $\sin x = \lambda\int_0^x e^{x-\xi}\, u(\xi)\, d\xi$ to the second kind and hence solve it.

Solution. The given equation is

$$\sin x = \lambda\int_0^x e^{x-\xi}\, u(\xi)\, d\xi \qquad (1)$$

Differentiating (1) w.r.t. *x*, we get

$$\cos x = \lambda\int_0^x e^{x-\xi}\, u(\xi)\, d\xi + \lambda e^{x-x} u(x).1$$

$$\Rightarrow \qquad \cos x = \lambda\int_0^x e^{x-\xi}\, u(\xi)\, d\xi + \lambda\, u(x)$$

$$\Rightarrow \qquad u(x) = \frac{1}{\lambda}\cos x - \int_0^x e^{x-\xi}\, u(\xi)\, d\xi \qquad (2)$$

which is Volterra integral equation of second kind and can be simply solved by the method of Laplace Transform.

1.6. Method of Iterated kernel/Resolvent kernel to solve the Volterra integral equation.

Consider the Volterra integral equation

$$u(x) = f(x) + \lambda \int_a^x K(x, \xi)\, u(\xi)\, d\xi \qquad (1)$$

We take $K_1(x, \xi) = K(x, \xi)$ (2)

and $$K_{n+1}(x, \xi) = \int_\xi^x K(x, t)\, K_n(t, \xi)\, dt \;;\; n = 1, 2, 3, \ldots \qquad (3)$$

From here, we get a sequence of new kernels and these kernels are called iterated kernels.

We know that (1) has one and only one series solution given by

$$u(x) = f(x) + \lambda \int_a^x K(x, t)\, f(t)\, dt + \lambda^2 \int_a^x \int_a^t K(x, t)\, K(t, t_1)\, f(t_1)\, dt_1 dt + \ldots \text{to } \infty \qquad (4)$$

We write this series solution in the form :

$$u(x) = u_0(x) + \lambda\, u_1(x) + \lambda^2\, u_2(x) + \ldots \text{ to } \infty \qquad (5)$$

Then comparing (4) and (5), we have

$$u_0(x) = f(x)$$

$$u_1(x) = \int_a^x K(x, t)\, f(t)\, dt = \int_a^x K_1(x, t)\, f(t)\, dt$$

and $$u_2(x) = \int_a^x \int_a^t K(x, t)\, K(t, t_1)\, f(t_1)\, dt_1\, dt$$

By interchanging the order of integration, we have

$$u_2(x) = \int_a^x f(t_1) \left[\int_{t_1}^x K(x, t)\, K_1(t, t_1)\, dt \right] dt_1$$

$$= \int_a^x f(t_1)\, K_2(x, t_1)\, dt_1 = \int_a^x f(t)\, K_2(x, t) dt$$

Similarly, $$u_n(x) = \int_a^x f(t)\, K_n(x, t)\, dt$$

Thus, (5) becomes

$$u(x) = f(x) + \lambda \int_a^x K_1(x, t)\, f(t)\, dt + \lambda^2 \int_a^x K_2(x, t)\, f(t)\, dt + \ldots \text{ to } \infty$$

$$\Rightarrow \qquad u(x) = f(x) + \lambda\int_a^x \left[K_1(x\,,\,t)+\lambda\,K_2(x\,,\,t)+\lambda^2\,K_3(x,t)+...\infty\right] f(t)\,dt$$

$$= f(x) + \lambda\int_a^x R(x\,,\,t\,:\,\lambda)\,f(t)\,dt \qquad (6)$$

where $R(x, t : \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x\,,\,t)$

Thus, (6) is the solution of given integral (1).

1.6.1. Neumann Series. The series $K_1 + \lambda K_2 + \lambda^2 K_3 + ...$ to ∞ is called the Neumann Series.

1.6.2. Resolvent Kernel. The sum of Neumann Series $R(x, t : \lambda)$ is called the Resolvent Kernel.

1.6.3. Example. With the aid of Resolvent Kernel find the solution of the integral equation

$$\phi(x) = x + \int_0^x (\xi - x)\,\phi(\xi)\,d\xi\,.$$

Solution. Here, $K_1(x, \xi) = K(x, \xi) = \xi - x$ (1)

and $K_{n+1}(x, \xi) = \int_\xi^x K(x\,,\,t)\,K_n(t\,,\,\xi)\,dt$ (2)

Putting n = 1, 2, 3,… in (2), we have,

$$K_2(x, \xi) = \int_\xi^x K(x\,,\,t)\,K_1(t\,,\,\xi)\,dt = \int_\xi^x (t-x)(\xi-t)\,dt = \frac{-1}{3!}(\xi-x)^3$$

and $$K_3(x, \xi) = \int_\xi^x K(x\,,\,t)\,K_2(t\,,\,\xi)\,dt = \int_\xi^x (t-x)\left[-\frac{1}{3!}(\xi-t)^3\right] dt = \frac{1}{5!}(\xi-t)^5$$

The Resolvent Kernel is defined as

$$R(x, \xi : \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x\,,\,\xi) = \frac{\xi-x}{1!} - \frac{(\xi-x)^3}{3!} + \frac{(\xi-x)^5}{5!}......... \text{ to } \infty \;(\lambda = 1)$$

$$= \sin(\xi - x)$$

The solution of the integral equation is given by

$$\phi(x) = f(x) + \lambda\int_0^x R(x\,,\,\xi\,:\,\lambda)\,f(\xi)\,d\xi$$

$$= x + \int_0^x \xi\,\sin(\xi - x)\,d\xi$$

$= x + \sin x - x$ [Integrating by parts]

$= \sin x$

This completes the solution.

1.6.4. Exercise. Obtaining the Resolvent Kernel, solve the following Volterra integral equation of second kind:

(1) $u(x) = f(x) + \lambda \int_0^x e^{x-\xi}\, u(\xi)\, d\xi$

Answer. $u(x) = f(x) + \lambda\, e^{(1+\lambda)x} \int_0^x e^{-(1+\lambda)\xi}\, f(\xi)\, d\xi$.

(2) $\phi(x) = 1 + \int_0^x (\xi - x)\, \phi(\xi)\, d\xi$

Answer. cos x.

(3) $\phi(x) = e^{x^2} + \int_0^x e^{x^2-\xi^2} \phi(\xi)\, d\xi$

Answer. $e^{x(x+1)}$.

1.7. Check Your Progress.

1. Reduce following initial value problem into Volterra integral equations:

$y'' - 2x\, y' - 3y = 0; \quad y(0) = 0,\ y'(0) = 0$

Answer. $y(x) = \int_0^x (x+\xi)\, y(\xi)\, d\xi$.

2. Using the method of successive approximation, solve the integral equation,

$u(x) = (1 + x) - \int_0^x u(\xi)\, d\xi$ with $u_0(x) = 1$.

Answer. 1.

3. Use the method of Laplace Transform to solve the following integral equations.

$u(x) = e^{-x} + \int_0^x \sin(x-\xi)\, u(\xi)\, d\xi$

Answer. $2e^{-x} - 1 + x$.

1.8. Summary. In this chapter, various methods like successive approximations, successive substitutions, resolvent kernel, Laplace transform are discussed to solve a Volterra integral equation. Also it is observed that a Volterra integral equation always transforms into an initial value problem.

Books Suggested:

1. Jerri, A.J., Introduction to Integral Equations with Applications, A Wiley-Interscience Publication, 1999.
2. Kanwal, R.P., Linear Integral Equations, Theory and Techniques, Academic Press, New York.
3. Lovitt, W.V., Linear Integral Equations, McGraw Hill, New York.
4. Hilderbrand, F.B., Methods of Applied Mathematics, Dover Publications.
5. Gelfand, J.M., Fomin, S.V., Calculus of Variations, Prentice Hall, New Jersey, 1963.